M303 Further Pure Mathematics

Exam 1 June 2016

Unofficial Solutions - these have NOT been checked by the module team. They will likely contain mistakes and omissions but may be helpful for revision purposes.

(Student version)

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Q 1.

(a)

Let $P(n)$ be the statement

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

For $n = 1$ we have $1(1+1) = \dfrac{1(1+1)(1+2)}{3}$

$$2 = \frac{1 \times 2 \times 3}{3}$$

So $P(1)$ is true and forms the basis of the induction.

Assume $P(k)$ is true for some $k \in \mathbb{N}$, then we want to show that $P(k+1)$ is true, that is

$$\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+1+1)(k+1+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$$

$$\begin{aligned} \textbf{LHS} = \sum_{i=1}^{k+1} i(i+1) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2), \text{ by assumption} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \quad = \textbf{RHS} \end{aligned}$$

So $P(k+1)$ is true and we have proved the statement $P(n)$ to be true by the Principle of Mathematical Induction HB p14.

(b)

(i) $715 = 3 \times 231 + 22$
$231 = 10 \times 22 + 11$
$22 = 2 \times 11 + 0$

11 is the last non-zero remainder so hcf(231,715) = 11

(ii) Running Euclid backwards

$$\begin{aligned} 11 &= 231 - 10 \times 22 \\ &= 231 - 10(715 - 3 \times 231) \\ &= -10 \times 715 + 31 \times 231 \end{aligned}$$

So, suitable integers are $s = 31$ and $t = -10$.

Q 2.

(a)

(i) $|G| = 12$.

(ii)

The maximum order of an element in $|G|$ is 6.

" Write down" so no reason needed, but Prop 2.10 HB p 32 applies.

(iii) Elements of $\mathbb{Z}_2$ can have order 1 or 2; elements of $\mathbb{Z}_6$ can have order 1, 2, 3 or 6.

So the only possibility is for an element of order 1 in $\mathbb{Z}_2$ and order 3 in $\mathbb{Z}_6$. There is one such element in $\mathbb{Z}_2$ and two such elements in $\mathbb{Z}_6$.

Hence there are two elements of order 3 in G.

(b)

(i) This is TRUE.

We know that $H \cap K$ is a subgroup of G. Take any $g \in G$. As H and K are normal in G then $gHg^{-1} \subseteq H$ and $gKg^{-1} \subseteq K$ (Prop. 3.7, HB p27). It follows that $g\,(H \cap K)\,g^{-1} \subseteq H \cap K$ and so by the same result, $H \cap K$ is normal in G.

(ii) This is FALSE.

A counterexample is given by $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which is Abelian and $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is a normal subgroup of G but is not cyclic.

(c) The homomorphism ϕ maps

$$\begin{aligned} 0 &\longmapsto 0 \\ 1 &\longmapsto 2 \\ 2 &\longmapsto 4 \\ 3 &\longmapsto 0 \\ 4 &\longmapsto 2 \\ 5 &\longmapsto 4 \end{aligned}$$

Therefore $\mathrm{Im}(\phi) = \{0, 2, 4\}$ and $\mathrm{Ker}(\phi) = \{0, 3\}$.

Q 3.

(a)

(i) $103 \equiv 7 \pmod{12} \equiv -5 \pmod{12}$

So by Prop. 4.7, HB p 45 $(3/103) = -1$.

(ii)

$$101 \equiv 1 \pmod 4 \text{ and } 103 \equiv 3 \pmod 4$$

So by Thm. 4.2, HB p 45 $(101/103) = (103/101)$.

Using Thm. 2.2 (a), HB p 45

$(103/101) = (2/101)$

And by Prop. 3.2, HB, p 45

$(2/101) = -1$, since $101 \equiv 5 \pmod 8$.

So $(101/103) = -1$.

(iii) $303 = 3 \times 101$ and so

$$\begin{aligned}(303/103) &= (3/103) \times (101/103) \text{ (Thm. 2.2, HB p45)} \\ &= -1 \times -1 \\ &= 1\end{aligned}$$

(b) Since $f(x)$ has degree 3, if it has any factors, one will be linear.

If $f(x) = x^3 + 17x^2 - 51x + 3$ has a rational root $\frac{p}{q}$, where hcf $(p, q) = 1$ then (by Thm. 4.7, HB p 51):

p must be an integer factor of 3, and

q must be an integer factor of 1.

So the options for $\frac{p}{q}$ are $\pm 1, \pm 3$. Now

$$\begin{aligned}f(1) &= 1 + 17 - 51 + 3 \neq 0 \\ f(-1) &= -1 + 17 + 51 + 3 \neq 0 \\ f(3) &= 27 + 17 \times 9 - 51 \times 3 + 3 \neq 0 \\ f(-3) &= -27 + 17 \times 9 + 51 \times 3 + 3 \neq 0\end{aligned}$$

Thus the options for a rational root are not possible and we conclude that $f(x)$ is irreducible over $\mathbb{Q}$.

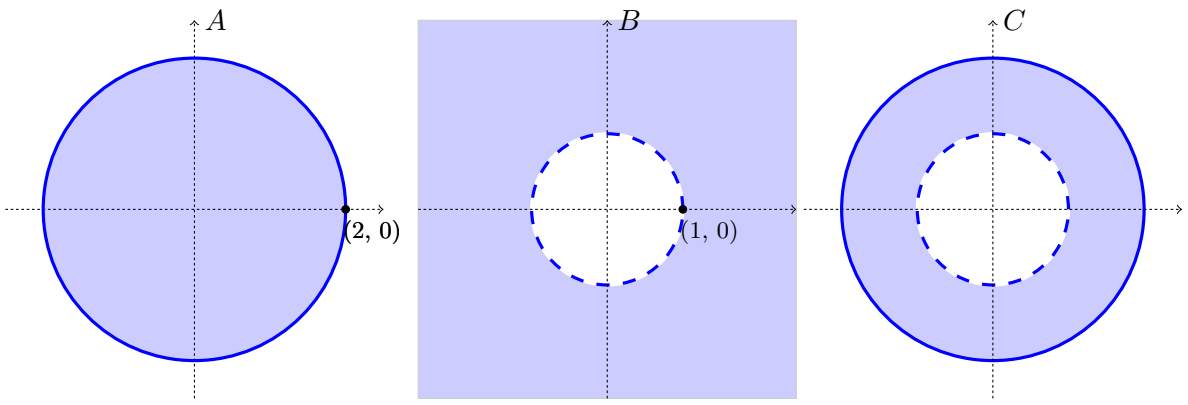
(c) 51 cannot be expressed as the sum of two squares. $51 = 3 \times 17$ and

$3 = 4k + 3$ (for $k = 0$).

So, by Prop. 4.13, HB p56, since 3 occurs to an odd power, 51 cannot be expressed as the sum of two squares.

Q 4.

(a)



(b)

(i)

$$\begin{aligned}
\mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) =& \{(a,b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4\},\\
\mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C) =& \{(a,b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 4\},\\
\mathrm{Bd}_{(\mathbb{R}^2,d^{(2)})}(C) =& \mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) - \mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C)\\
=& \{(a,b) \in \mathbb{R}^2 \colon a^2 + b^2 = 1\} \cup \{(a,b) \in \mathbb{R}^2 \colon a^2 + b^2 = 4\}
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathrm{Cl}_{(\mathbb{R}^2,d_0)}(C) =& C,\\
\mathrm{Int}_{(\mathbb{R}^2,d_0)}(C) =& C,\\
\mathrm{Bd}_{(\mathbb{R}^2,d_0)}(C) =& \mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) - \mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C) = \emptyset.
\end{aligned}$$

(c) C is bounded but not closed and as it is subset of $\mathbb{R}^2$ it is not compact by the Heine-Borel Theorem HB page 134.

Q 5.

$\alpha = \sqrt{5} - \sqrt{2}$.

Given: $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2, \sqrt{2} \notin \mathbb{Q}(\sqrt{5}), \sqrt{5} \notin \mathbb{Q}(\sqrt{2})$

(a)

$$\begin{aligned}\alpha^3 &= (\sqrt{5} - \sqrt{2})^2(\sqrt{5} - \sqrt{2}) \\ &= (7 - 2\sqrt{2}\sqrt{5})(\sqrt{5} - \sqrt{2}) \\ &= (7\sqrt{5} - 7\sqrt{2} - 10\sqrt{2} + 4\sqrt{5}) \\ &= 11\sqrt{5} - 17\sqrt{2}\end{aligned}$$

Hence $a = -17, b = 11$.

(b)

$$\begin{aligned}\alpha^3 - 11\alpha &= 11\sqrt{5} - 17\sqrt{2} - 11(\sqrt{5} - \sqrt{2}) \in \mathbb{Q}(\alpha) \\ &= -6\sqrt{2} \in \mathbb{Q}(\alpha)\end{aligned}$$

and $\sqrt{2} \in \mathbb{Q}(\alpha)$. $\sqrt{5} = \alpha + \sqrt{2}$ so $\sqrt{5} \in \mathbb{Q}(\alpha)$.

(c) Since $\alpha \in \mathbb{Q}(\sqrt{5}, \sqrt{2})$
and, from (b) $\mathbb{Q}(\sqrt{5}, \sqrt{2}) \subseteq \mathbb{Q}(\alpha)$
It follows that $\mathbb{Q}(\sqrt{5}, \sqrt{2}) = \mathbb{Q}(\alpha)$.

(d) Since $\sqrt{2} \notin \mathbb{Q}(\sqrt{5})$ and $\sqrt{2}$ is a root of $x^2 - 2$, a quadratic polynomial over $\mathbb{Q}(\sqrt{5})$, $[\mathbb{Q}(\sqrt{2}, \sqrt{5}) : \mathbb{Q}(\sqrt{5})] = 2$.

By the KLM Theorem (HBp105),

$$\begin{aligned}[\mathbb{Q}(\alpha) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2}, \sqrt{5}) : \mathbb{Q}(\sqrt{5})] \times [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] \\ &= 2 \times 2 = 4\end{aligned}$$

(e) $\alpha^2 = 7 - 2\sqrt{2}\sqrt{5}$
$\implies 7 - \alpha^2 = 2\sqrt{2}\sqrt{5}$
$\implies (7 - \alpha^2)^2 = (2\sqrt{2}\sqrt{5})^2$
$\implies \alpha^4 - 14\alpha^2 + 49 = 40 \implies \alpha^4 - 14\alpha^2 + 9 = 0$

Minimal polynomial must have degree 4 (from d),
so $x^4 - 14x^2 + 9$ is the required polynomial.

Q 6.

(a)

$d((3,7),1,2)) = \max\{|3-1|, |7-2|\} = 5$

(b)

Consider the sets

$$U = \{(x,y) \in \mathbb{R}^2 : y < 1\}, \quad V = \{(x,y) \in \mathbb{R}^2 : y > 1\}.$$

Then $U \cup V = X$ and $U \cap V = \emptyset$, and U and V are both non-empty and simultaneously d_X-closed and d_X-open, then by Definition 3.1 HB p120, U and V form a disconnection of X.

(c)

Consider $d(\mathbf{a}_n, (0,0)) = \max\{|\cos(n)/n - 0|, |\sin(n)/n - 0|\}$

As $|\cos(n)| \leq 1$ and $|\sin(n)| \leq 1$ then $|\cos(n)/n| \leq 1/n$ and $|\sin(n)/n| \leq 1/n$

As $(1/n)$ is a basic null sequence by Theorem 1.13 HB p63 then

$d(\mathbf{a}_n, (0,0)) \longrightarrow 0$ as $n \longrightarrow \infty$ and so $(\mathbf{a}_n)$ is a d-convergent sequence as required.

(d)

Consider the sequence $(\mathbf{b}_n) = (0, 1 + 1/n)$

then $d(\mathbf{b}_n, (0,1)) = \max\{|0-0|, |1+1/n-1|\} = 1/n$

As $(1/n)$ is a basic null sequence as noted above then

$d(\mathbf{b}_n, (0,1)) \longrightarrow 0$ as $n \longrightarrow \infty$ and so $(\mathbf{b}_n)$ is a convergent sequence with limit $(0,1)$. Therefore by Lemma 1.2 and Definition 1.3 HB p135 then $(\mathbf{b}_n)$ is a Cauchy sequence in X that converges to $(0,1) \notin X$ and so by Definition 2.1 HB p136 X is not d-complete.

(e)

From Theorem 2.3 HB p136 if X were compact it would be complete and so from the result in (d) above it is not compact. As X is not compact then by Theorem 5.10 HB p133 it cannot be sequentially compact.

Q 7.

(a)

By the Sylow Theorems HB, p39, G has n_q Sylow q-sugroups of order q,
where $n_q \equiv 1 \,(\mathrm{mod}\, q)$ and $n_q | \, 9$. As q is a prime greater than or equal to 5 then $n_q = 1$ as $9 \not\equiv 1 \,(\mathrm{mod}\, 5)$ and $9 \not\equiv 1 \,(\mathrm{mod}\, 7)$ and all other primes are greater than 9. Hence G has a unique Sylow q-subgroup.

(b)

(i)

By the Sylow Theorems HB, p39, G has n_3 Sylow 3-subgroups of order 9,
where $n_3 \equiv 1 \,(\mathrm{mod}\, 3)$ and $n_3 | \, 11$. Hence, as $11 \not\equiv 1 \,(\mathrm{mod}\, 3)$, G has a unique Sylow 3-subgroup of order 9.

(ii)

We know from (a) and (b)(i) above that if H is the unique Sylow 3-subgroup of order 9 and K is the unique Sylow 11-subgroup of order 11 then $|H| \times |K| = 9 \times 11 = 99 = |G|$. Further 9 and 11 are co-prime and as both H and K are unique subgroups of their order then, by Lemma 1.9 HBp39, they are normal. Hence, by Theorem 2.3 HBp39, G is the internal direct product of H and K. Also, as $|H| = 9$ and $|K| = 11$, $H \cong \mathbb{Z}_9$ or $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $K \cong \mathbb{Z}_{11}$. So both H and K are abelian and hence G is also Abelian.

(iii)

The Sylow 3-subgroup of order 9 is isomorphic either to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ by Theorem 3.2 HBp36. Therefore all groups of order 99 are isomorphic either to $\mathbb{Z}_{99} \cong \mathbb{Z}_9 \times \mathbb{Z}_{11}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \cong \mathbb{Z}_3 \times \mathbb{Z}_{33}$.

(c)

The argument does not generalise. Consider for instance the case $q = 7 \equiv 1 \,(\mathrm{mod}\, 3)$. We know from part (b)(i) that G could have 1 or 7 Sylow 3-subgroups. As there may be seven Sylow 3-subgroups, Lemma 1.9 HB, p39 shows that, in that event, there is no normal subgroup of order 9, and the argument breaks down.

(d)

If $p = 2$, $r = 3$, then $p^2 r = 4 \times 3 = 12$. Dic_3 is a group of order 12 that is not Abelian.

Q 8.

(a) If $N(r) = 2$, then $a^2 + b^2 = 2$ for some integers a, $b < 2$, as $2^2 = 4 > 2$. If $a = 0$ and $b = 0$, the result is trivial. WLOG let $a = 0$. then $b^2 = 2$ but this has no integer solution. Hence the only possibility is that $a^2 = 1$ and $b^2 = 1$ and so $a = \pm 1$ and $b = \pm 1$. The only elements $r \in \mathbb{Z}[i]$ for which $N(r) = 2$ are therefore $1 + i$, $1 - i$, $-1 + i$ and $-1 - i$.

(b) We first note that, being the square of the complex modulus, the given norm is multiplicative. (Definition 2.14 HB, p53).
Now suppose that $s = rp$ and $t = rq$ for some p,$q \in \mathbb{Z}[i]$. As N is multiplicative, $N(s) = N(rp) = N(r)N(p)$ and $N(t) = N(rq) = N(r)N(q)$.
Thus $N(r)$ is a common factor of $N(s)$ and $N(t)$ in $\mathbb{Z}$.

(c) We have $N(3 + i) = 3^2 + 1 = 10 = 2 \times 5$ and $N(6) = 36 = 2^2 \times 3^2$. So by part (b) any factor of $3 + i$ and 6 must have norm 1 or 2. Any factor with norm 1 will be a unit as $\mathbb{Z}[i]$ is a Euclidean Domain (Lemma 3.5(b) HBp54) and therefore, as $N(1 + i) = 2$, $1 + i$ is a highest common factor of $3 + i$ and 6 in $\mathbb{Z}[i]$. As $1 - i$, $-1 + i$, $-1 - i$ are associates of $1 + i$, they are also highest common factors of $3 + i$ and 6 in $\mathbb{Z}[i]$.

(d) As $\mathbb{Z}[i]$ is a PID, $I = \langle 6, 3 + i\rangle = \langle \mathrm{hcf}(6, 3 + i)\rangle$ by Lemma 3.7 HBp97. From part (c), we know that the highest common factors of 6 and $3 + i$ are $1 + i$, $1 - i$, $-1 + i$ and $-1 - i$. We also know that $N(\mathrm{hcf}(6, 3 + i)) = 2$ and that the only elements $r \in \mathbb{Z}[i]$ such that $N(r) = 2$ are $1 + i$, $1 - i$, $-1 + i$ and $-1 - i$, from parts (b) and (a) respectively. It follows that $\{1 + i, 1 - i, -1 + i, -1 - i\}$ is the set of elements $d \in \mathbb{Z}[i]$ such that $I = \langle d\rangle$.

(e) From part (d),

$$I = \langle 1 + i\rangle = \{r(1 + i) : r \in \mathbb{Z}[i]\} = \{r + ri : r \in \mathbb{Z}[i]\}.$$

Putting $r = c + di$, $c, d \in \mathbb{Z}$, we get

$$I = \{c + di + ci - d : c, d \in \mathbb{Z}\} = \{(c - d) + (c + d)i : c, d \in \mathbb{Z}\}.$$

Now, since $d \equiv -d \pmod 2$, we have $c - d \equiv c + d \pmod 2$.
Therefore $I \subseteq \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha \equiv \beta \pmod 2\}$.
Conversely, if $\alpha \equiv \beta \pmod 2$, then $c = (\alpha + \beta)/2 \in \mathbb{Z}$ and $d = (\beta - \alpha)/2 \in \mathbb{Z}$. Thus $\alpha + \beta i = (c - d) + (c + d)i$ for $c, d \in \mathbb{Z}$. Hence $\{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha \equiv \beta \pmod 2\} \subseteq I$, completing the proof that $I = \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha \equiv \beta \pmod 2\}$.

(f) As $N(1 + i) = 2$ is prime in $\mathbb{Z}$, $1 + i$ is irreducible in $\mathbb{Z}[i]$. Since, if $1 + i = ab$, $a, b \in \mathbb{Z}[i]$, then $N(a)N(b) = 2$, that is WLOG $N(a) = 2$ and $N(b) = 1$. So b is a unit in $\mathbb{Z}[i]$ by Lemma 3.5(b) HBp54. Hence, as $1 + i$ is a non-zero non-unit element of the PID $\mathbb{Z}[i]$, $I = \langle 1 + i\rangle$ is a maximal ideal of $\mathbb{Z}[i]$ by Theorem 4.22 HBp99.

Q 9.

(a)

(i) $f_n(0) = \frac{1}{n}\left(|\sin(0)| + |\cos(0)|\right) = \frac{1}{n}(0+1) = \frac{1}{n}$

$f_n(1) = \frac{1}{n}\left(|\sin\left(\frac{\pi}{2}\right)| + |\cos\left(\frac{\pi}{2}\right)|\right) = \frac{1}{n}(1+0) = \frac{1}{n}$

(ii) We must show that $(d_{\max}(f_n,\ f_0))$ is a null sequence. Now, by definition,

$$d_{\max}(f_n,\ f_0) \quad = \quad \max\left\{|f_n(x) - f_0(x)| \ :\ x \in [0,\ 1]\right\}.$$

Also,

$$\begin{aligned} |f_n(x) - f_0(x)| &= |f_n(x)| \\ &= \frac{1}{n}\left(|\sin\left(\frac{\pi}{2}x\right)| + |\cos\left(\frac{\pi}{2}x\right)|\right) \\ &\leq \frac{1}{n}(1+1) = \frac{2}{n} \end{aligned}$$

since $|\sin\left(\frac{\pi}{2}x\right)| \leq 1$ and $|\cos\left(\frac{\pi}{2}x\right)| \leq 1$.

So,

$$d_{\max}(f_n,\ f_0) \leq \frac{2}{n}$$

The sequence $\left(\frac{1}{n}\right)$ is a Basic Null Sequence and so, by the Combination Rules, $\left(\frac{2}{n}\right)$ is a null sequence and hence, by the Squeeze Rule, $(d_{\max}(f_n,\ f_0))$ is a null sequence.

Therefore, (f_n) is $d_{\max}$-convergent to f_0.

(iii) To show that $f_n \in S$, we must show that $f_n(x) > 0$ for all $x \in [0,\ 1]$.

If $x = 0$ then, by part (a)(i), $f_n(x) = f_n(0) = \frac{1}{n} > 0$.

If $x \in (0, 1]$ then $|\sin\left(\frac{\pi}{2}x\right)| > 0$ so $f_n(x) > 0$.

Hence, for all $x \in [0, 1]$, $f_n(x) > 0$ and so $f_n \in S$ for all $n \in \mathbb{N}$.

(iv) Now, $f \in S \Rightarrow f(x) > 0$ for each $x \in [0,\ 1]$, so $f_0 \notin S$.

By parts (ii) and (iii), (f_n) is a sequence in S that is $d_{\max}$-convergent to f_0 which is not in S, so S is not $d_{\max}$-closed.

However, S is a subset of $C[0,\ 1]$ which is $d_{\max}$-complete by Theorem 2.8, HB p136.

So, $(S,\ d_{\max})$ is not complete by the contrapositive of Proposition 2.2.1, HB p136.

Hence, S is not $d_{\max}$-compact, by the contrapositive of Theorem 2.3, HB p 136.

(b)

(i)

Let (f_n) be a sequence in A such that f_n $d_{\max}$-converges to f. We must show that $f \in A$. That is, we must show that,

$$|f(x) - f(y)| \leq 2|x - y|, \quad \text{for each } x,\ y \in [0,\ 1]$$

Now, by the Triangle Inequality, for each x, $y \in [0, 1]$

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq d_{\max}(f_n, f) + 2|x - y| + d_{\max}(f_n, f) \end{aligned}$$

since $f_n \in A$. This inequality is true for arbitrary n and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq 0 + 2|x - y| + 0, \text{ for each} x, y \in [0, 1]$$

and so $f \in A$. Hence A is $d_{\max}$-closed.

(ii)

By the definition of the set B

$$f \in B \iff |f(x) - 0| \leq 2, \ for\ all\ x \in [0, 1] \iff d_{max}(f, f_0) \leq 2$$

so $B = B_{d_{max}}[f_0, 2]$ which is a closed ball and therefore a closed set in $(C[0, 1], d_{max})$ by Appendix 5, HB p89.

Thus A and B are both $d_{\max}$-closed and so their intersection $A \cap B$ is $d_{\max}$-closed by Proposition 1.17, HB p81.

(iii) Since $A \cap B \subseteq C[0, 1]$, we use the Arzelà-Ascoli Theorem to show that $A \cap B$ is $d_{\max}$-sequentially compact.

1. From part (ii) above, $A \cap B$ is $d_{\max}$-closed

2. From example 1. of pointwise bounded sets with $g(x) = 2$, HB p130, B is a pointwise bounded set and so $A \cap B \subseteq B$ is also a pointwise bounded set of functions.

3. For all $f \in A$, $|f(x) - f(y)| \leq 2|x - y|$, so f is a Lipschitz function with Lipschitz constant 2. So A is an equicontinuous set of functions by example 1., HB p131. Therefore, $A \cap B \subseteq A$ is also an equicontinuous set of functions by example 2., HB p131.

Thus all three conditions of the Arzelà-Ascoli Theorem are satisfied and so $A \cap B$ is $d_{\max}$-sequentially compact.